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# *TEMPERAMENT; OR, THE DIVISION OF THE OCTAVE.*

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PART I.



## 1. Introduction.

The investigations which form the subject of the present communication had their origin in a practical source. It is conceived important that this should be understood, as many musicians regard the problem as a purely theoretical one, undeserving of practical attention. It was in taking part in the tuning of an organ that the effect of the ordinary equal temperament was first realised by the writer as a matter affecting musical sounds in practice, as distinguished from theory. It is the writer's experience that after the ear has once been attracted to this effect, it never fails to perceive it in the tones of instruments tuned in the ordinary way. Some tones show the effect more, some show it less. Among keyed instruments the worst effects are produced by the ordinary harmonium; next to this comes the full-toned modern grand piano, which is often unpleasant in slow harmony, especially if, as is generally the case, the temperament is not very uniform; * and the best tones, which show

* The chord of E major is constantly found 'rank' on these pianos. This would suggest some defect in the process of tuning ordinarily used.

the effect least, are those of soft-toned pianos with little power, and the ordinary organ diapason-stops—especially those old-fashioned, sweet-toned diapasons, which are rapidly disappearing before the organ-builders of the present day. The constant perception of these effects served as an inducement to a study of the subject, pursued in the first instance with the practical object of the improvement of instruments, and afterwards also for the sake of the interest attaching to the theory developed.

The problem to be dealt with is well stated as follows, in the Preface to 'A Theory of Harmony, founded on the Tempered Scale,' by Dr. Stainer:—'When musical mathematicians shall have agreed amongst themselves upon the exact number of divisions necessary in the octave; when mechanists shall have constructed instruments upon which the new scale can be played; when practical musicians shall have framed a new notation, which shall point out to the performer the ratio of the note he is to sound to the generator; when genius shall have used all this new material to the glory of art—then it will be time enough to found a new theory of harmony on a mathematical basis.'

This passage was of considerable use in directing attention to the points of importance in theory; it contains, however, some confusion of ideas, as will be pointed out immediately.

In the first place, before any conclusion can be come to as to the number of divisions necessary in the octave, it is clear that the theory of the division of the octave must be studied in a more complete and comprehensive manner than has been usual in the theory of music. In fact, when we come to examine the subject, we shall find that, although the properties of isolated systems have been studied here and there, yet no comprehensive method has been given for the derivation and treatment of such systems;* and the establishment of such a method will be the first point which will demand our attention. We shall then come to the conclusion that different systems have their different advantages; and we may contemplate the possibility of music being written for, or adapted to, one system or another, just as hitherto music has been written for performance in one key or another of the equal temperament.

In the second place, notation will be provided, by which the exact note intended to be played can be indicated to the performer, in those systems in which a modification of the ordinary notation is necessary. The notation is so constructed as to supplement the ordinary notation without altering it, and the signs required in addition to the ordinary notation are few in number and simple in their character.

Again, the problem of instruments has been solved in a general manner as far as keyed instruments are concerned. A generalised keyboard has been devised, by means of which it is

* See, however, Mr. A. J. Ellis on the 'Temperament of Musical Instruments with Fixed Tones.'—Royal Society's Proceedings, 1864.

possible to control the notes of all systems which proceed by continuous series of equal fifths; and this keyboard has been constructed and applied.

The confusion of ideas above alluded to arises from the assumption that the theory to be employed will be based on the derivation of scales from some one harmonic series, as well as from the division of the octave. Now this will never be true. If scales are derived from the division of the octave, their notes can never be more than approximations to the notes of any one harmonic series. In some of the systems subsequently developed, conditions having reference to the properties of harmonics will be employed; for example, we may make our fifths or our thirds perfect. This class of conditions is regarded as being derived from the harmonic series of each pair of notes employed. But the notation employed has not in any case reference to the ratio of the note sounded to any generator.

In the present introductory paper, the principal properties of the class of systems dealt with will be established in a general manner, the notation above referred to will be explained, and a brief account will be given of what may be called the 'principle of symmetrical arrangement.' This principle is the foundation of the arrangement of the keyboard above referred to; and its chief characteristic is, that any given interval, or combination of intervals, presents the same form on such an arrangement, on whatever notes it is taken—whence the form of fingering on the keyboard is the same in all keys.

## 2. Expression of Intervals.

Before commencing the treatment of these subjects, it will, however, be necessary to make some remarks on the method employed for the expression and calculation of intervals.

All intervals will be expressed in terms of equal temperament semitones. The letters E. T. will be used as an abbreviation for the words 'equal temperament.' Thus an octave, which is 12 E. T. semitones, will be written as 12; the 53rd part of an octave will be written as $\frac{12}{53}$, or ·22642. Five places of decimals will be generally considered sufficient.

The following rules for transforming vibrations ratios into the equivalent E. T. interval can be made use of by any one who knows how to look out a logarithm in a table. They obviously depend on the form of log. 2 :—

Rule I.—To find the equivalent of a given vibrations ratio in E. T. semitones. Take the common logarithm of the given ratio, subtract $\frac{1}{300}$, and call this the first improved value (F. I. V.). From the original logarithm subtract $\frac{1}{300}$ of the first improved value, and $\frac{1}{10,000}$ of the first improved value. Multiply the remainder by 40. The result is the interval, expressed in E. T. semitones.

EXAMPLE 1.—To find the values of a perfect fifth, the vibrations ratio of which is $\frac{3}{2}$ in E. T. semitones:—

| | | | |
|---|---|---|---|
| Log. 3 | = | ·4771213 | ·1760213 |
| Log. 2 | = | ·3010300 | 5850 |
| Log. $\frac{3}{2}$ | = | ·1760913 | ·1755063 |
| $\frac{1}{300}$ | = | ·0005870 | 175 |
| F. I. V. | = | ·1755043 | ·1754888 |
| | | | 40 |
| | | | 7·01955\|2 |

Thus a perfect fifth exceeds 7 semitones by ·01955.

N.B.—The rule only gives five places correct.

The true value of the perfect fifth calculated by an exact process to 20 places is:—

7·01955 00086 53874 17740

EXAMPLE 2.—To find the value of a perfect third, the vibrations ratio of which is $\frac{5}{4}$, in E. T. semitones:—

| | | | |
|---|---|---|---|
| Log. 5 | = | ·6989700 | ·0969100 |
| Log. 4 | = | ·6020600 | 3219 |
| Log. $\frac{5}{4}$ | = | ·0969100 | ·0965881 |
| $\frac{1}{300}$ | = | ·0003230 | 96 |
| | | ·0965870 | ·0965785 |
| | | | 40 |
| | | | 3·86314\|0 = 4 − ·13686 |

Thus a perfect third exceeds 3 semitones by ·86314; or, as it is generally more convenient to state the result, it falls short of 4 semitones by ·13686.

The value to 20 places is:—

4 − ·13686 28613 51651 82551

RULE II.—To find the vibrations ratio of an interval given in E. T. semitones.

To the given number add $\frac{1}{300}$ and $\frac{1}{10000}$ of itself. Divide by 40. The result is the logarithm of the ratio required.

EXAMPLE.—The E. T. third is 4 semitones. The vibrations ratio, found as above, is 1·25992. Hence the vibrations ratio of the E. T. third to the perfect third is nearly 126 : 125.

## 3. DEFINITIONS.

*Regular Systems* are such that all their notes can be arranged in a continuous series of equal fifths.

*Regular Cyclical Systems* are not only regular, but return into the same pitch after a certain number of fifths: every such system divides the octave into a certain number of equal intervals.

*Error* is deviation from a perfect interval.

*Departure* is deviation from an E. T. interval.

Intervals taken upwards are called positive, taken downwards negative.

Fifths are called positive if they have positive departures, that is, if they are greater than E T. fifths; they are called negative if they have negative departure, that is, if they are less than E. T. fifths. We saw that perfect fifths are more than 7 semitones; they are therefore positive.

Systems are said to be positive or negative according as their fifths are positive or negative.

Regular Cyclical Systems are said to be of the *r*th order, positive or negative, when the departure of 12 fifths is $\pm r$ units of the system.

Thus, we shall see later that in the system of 53 the departure of 12 fifths is 1 unit upwards, and the system is positive of the first order; in the system of 118 the departure of 12 fifths is 2 units upwards, and the system is said to be positive of the second order. In the system of 31 the departure of 12 fifths is one unit downwards, and the system is negative of the first order; and in the system of 50 the departure of 12 fifths is two units downwards, and the system is negative of the second order. Systems of the first order are called Primary, of the second order Secondary.

### 4. Formation of Intervals by Series of Fifths.

When successions of fifths are spoken of, it is intended that octaves be disregarded. If the result of a number of fifths is expressed in E. T. semitones, any multiples of 12 (octaves) may be subtracted or added.

As an example, we will consider some of the intervals formed by successive fifths in the system of 53. We shall see later (Theorem iv.) that the fifth of this system is $7\frac{1}{53}$; *i.e.*, it exceeds the equal temperament fifth by $\frac{1}{53}$ of an E. T. semitone. This being premised, we have the following intervals, amongst others:—

Departure of 12 fifths $= \frac{12}{53}$.

For $12 \times 7\frac{1}{53} = 84\frac{12}{53}$; and we subtract 84, which represents 7 octaves.

Two-fifths tone $= 2\frac{2}{53}$.

For $2 \times 7\frac{1}{53} = 14\frac{2}{53}$; and we subtract 12, which represents 1 octave.

* Seven-fifths semitone, formed by 7 fifths up, $= 1\frac{7}{53}$.

For $7 \times 7\frac{1}{53} = 49\frac{7}{53}$; and we subtract 48, which represents 4 octaves.

* Five-fifths semitone, formed by 5 fifths down, $= 1 - \frac{5}{53}$

For $5 \times - 7\frac{1}{53} = - 35\frac{5}{53}$; and we add 36, which represents 3 octaves.

* These expressions were suggested to the writer by Mr. Parratt.

Or, if we consider the system of 31 in which the fifth is $7 - \frac{1}{31}$, we have, similarly:

Departure of 12 fifths $= \quad -\frac{12}{31}$
Two-fifths tone $= 2 - \frac{2}{31}$
Seven-fifths semitone $= 1 - \frac{7}{31}$
Five-fifths semitone $= 1\frac{5}{31}$

## 5. Regular Systems.

The importance of Regular Systems arises from the symmetry of the scales which they form.

*Theorem* i.—In any regular system 5 seven-fifths semitones and 7 five-fifths semitones make up an exact octave.

For the departures from E. T. of the 5 seven-fifths semitones are due to 35 fifths up, and the departures of the 7 five-fifths semitones are due to 35 fifths down, leaving 12 E. T. semitones, which form an exact octave.

Example.—A perfect fifth $= 7 + \delta$, where $\delta = \cdot 01955$.
Then the seven-fifths semitone is $1 + 7\,\delta$,
the five-fifths semitone is $1 - 5\,\delta$,
and 5 seven-fifths semitones, together with 7 five-fifths semitones, is:—

$$5\,(1 + 7\,\delta) + 7\,(1 - 5\,\delta) = 5 + 35\,\delta + 7 - 35\,\delta = 12.$$

*Theorem* ii.—In any regular system the difference between the seven-fifths semitone and the five-fifths semitone is the departure of 12 fifths, having regard to sign.

That is to say, if we subtract the five-fifths semitone from the seven-fifths semitone, the result is equal to the departure of 12 fifths in value; and it is positive if the fifths are positive, and negative if the fifths are negative.

For the seven-fifths semitone up is one E. T. semitone up and the departure of 7 fifths up, and the five-fifths semitone down is one E. T. semitone down and the departure of 5 more fifths up—which makes, on the whole, the departure of 12 fifths up, and if the single departures are positive, then the twelve departures are positive, and if negative, negative.

Example 1.—A perfect fifth $= 7 + \delta$ as before, and $\delta$ is positive.
And seven-fifths semitone $= 1 + 7\,\delta$
five-fifths semitone $= 1 - 5\,\delta$,
whence, subtracting the lower line, the difference $= 12\,\delta$.

Example 2.—A fifth of the system of 31 $= 7 - \frac{1}{31}$, and it is negative.
The seven-fifths semitone $= 1 - \frac{7}{31}$
Five-fifths semitone $= 1 + \frac{5}{31}$
whence, subtracting the lower line, the difference $= -\frac{12}{31}$.

## 6. Regular Cyclical Systems.

The importance of Regular Cyclical Systems arises from the infinite freedom of modulation in every direction which is possible in such systems when properly arranged; whereas in non-cyclical systems required modulations are liable to be impossible, owing to the demand arising for notes outside the material provided.

*Theorem* iii.—In a Regular Cyclical System of order $\pm r$, the difference between the seven-fifths semitone and five-fifths semitone is $\pm r$ units of the system.

For, recalling the definition of $r$th order (departure of 12 fifths $= \pm r$ units), the proposition follows from Theorem ii.

EXAMPLE 1.—In the system of 53 the fifth is $7\frac{1}{53}$;

Seven-fifths semitone $= 1\frac{7}{53}$

Five-fifths semitone $= 1 - \frac{5}{53}$,

whence subtracting, the difference is $\frac{12}{53}$, which is the octave divided by 53, or one unit of the system.

EXAMPLE 2.—In the system of 31 the fifth is $7 - \frac{1}{31}$, and, as before, the difference is $- \frac{12}{31}$, or $-$ (one unit of the system).

*Corollary.*—This proposition, taken with Theorem i., enables us to determine the numbers of divisions in the octave in systems of any order, by introducing the consideration that each semitone must consist of an integral number of units. The principal known systems are here enumerated:—

| 7-fifths semitone $= x$ units | 5-fifths semitone $= y$ units | Number of Units in octave (Th. i.) $5x + 7y = n$ |
|---|---|---|
| | Primary (1st order) Positive. | |
| 2 | 1 | 17 |
| 3 | 2 | 29 |
| 4 | 3 | 41 |
| 5 | 4 | 53 |
| 6 | 5 | 65 |
| | Secondary (2nd order) Positive. | |
| 11 | 9 | 118 |
| | Primary Negative. | |
| 1 | 2 | 19 |
| 2 | 3 | 31 |
| | Secondary Negative. | |
| 3 | 5 | 50 |

The mode of formation in other cases is obvious.

Passing over, for the present, the derivation of scales from this scheme, we proceed to other important theorems on Cyclical Systems:—

*Theorem* iv.—In any Regular Cyclical System, if the octave be divided into $n$ equal intervals, and $r$ be the order of the system, the departure of each fifth of the system is $\frac{r}{n}$ E. T. semitones.

Let the departure of each fifth of the system be $\delta$. Then the departure of twelve fifths $= 12\,\delta = r$ units by definition, and the unit $= \frac{12}{n}$ E. T. semitones (since the octave, which is 12 semitones, is divided into $n$ equal parts). Hence

$$12\,\delta = r.\frac{12}{n} \text{ or } \delta = \frac{r}{n}$$

EXAMPLE 1.—In the system of 53, which is of the first order and positive (Th. iii. Cor.), the departure of 12 fifths = 1 unit, $= \frac{12}{53}$; whence the departure of one fifth $= \frac{1}{53}$.

EXAMPLE 2.—In the system of 118, which is of the second order and positive, the departure of 12 fifths is 2 units, $= 2.\frac{12}{118}$; whence the departure of one fifth is $\frac{2}{118}$, or $\frac{1}{59}$.

EXAMPLE 3.—In the system of 31, which is of the first order and negative, the departure of each fifth is $- \frac{1}{31}$.

EXAMPLE 4.—In the system of 50, which is of the second order and negative, the departure of each fifth is $- \frac{1}{25}$.

*Theorem* v.—If, in a system of the $r$th order, the octave be divided into $n$ equal intervals, $r + 7n$ is a multiple of 12, and $\frac{r+7n}{12}$ is the number of units in the fifth of the system.

Let $\phi$ be the number of units in the fifth.

Then $\phi . \frac{12}{n}$ is the fifth, and $=7+\delta$, if $\delta$ be the departure of the fifth; $=7+\frac{r}{n}$ by Th. iv.

Hence $\phi=\frac{7n+r}{12}$,

and $\phi$ is an integer by hypothesis—whence the proposition.

From this proposition we can deduce corresponding values of $n$ and $r$. This is useful in investigating systems of the higher orders. Casting out multiples of 12, where necessary, from $n$ and $r$, we have the following relations between the remainders:—

Remainder of

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| $r$ | 5 | 10 | 3 | 8 | 1 | 6 | 11 | 4 | 9 | 2 | 7 |
| | −7 | −2 | −9 | −4 | −11 | −6 | −1 | −8 | −3 | −10 | −5 |

EXAMPLE.—It is required to find the order of the system in which the octave is divided into 301 equal intervals.

300 is a multiple of 12; remainder 1 gives order 5, or − 7. 301 is a system of some interest regarded as a positive system of order 5, in consequence of its having, as we shall see later, tolerably good fifths and thirds; while its intervals are expressed by the first three places of the logarithms of the vibration ratios, ·3010 being the first four places of log. 2. Mr. Ellis has recently made use of this system.—(Royal Society's Proceedings, 1874.)

*Theorem* vi.—If a system divide the octave into $n$ equal intervals, the total departure of all the $n$ fifths of the system $=r$ E. T. semitones, where $r$ is the order of the system.

For, if $\delta$ be the departure of one fifth, then, by Th. iv.,

$$\delta = \frac{r}{n}; \text{ whence } n\,\delta = r.$$

or the departure of $n$ fifths $= r$ semitones.

EXAMPLE 1.—The departure of 53 fifths of the system of 53 is 1 semitone; for the departure of one fifth is $\frac{1}{53}$ by Th. iv.

EXAMPLE 2.—The departure of 118 fifths of the system of 118 is 2 semitones; for the departure of one-fifth is $\frac{2}{118}$.

EXAMPLE 3.—The departure of 31 fifths of the system of 31 is $-$ 1 semitone (one semitone flat); for the departure of one-fifth is $-\frac{1}{31}$.

EXAMPLE 4.—The departure of 50 fifths of the system of 50 is $-$ 2 semitones.

This theorem gives rise to a curious mode of deriving the different systems.

Suppose the notes of an E. T. series arranged, on a horizontal line, in the order of a succession of fifths, and proceeding onwards indefinitely, thus:

*c g d a e b f♯ c♯ g♯ d♯ a♯ f c g . . . .*

and so on.

Let a regular system of fifths start from *c*. If they are positive, then at each step the pitch rises farther from E. T. It can only return to *c* by sharpening an E. T. note.

Suppose that *b* is sharpened one E. T. semitone, so as to become *c*; then the return may be effected—

at the first *b* in 5 fifths
—— second *b* in 17 fifths
—— third *b* in 29 fifths,—

and so on. Thus we obtain the primary positive systems.

Secondary positive systems may be got by sharpening *b♭* by two semitones, and so on. If the fifths are negative, the return may be effected by depressing *c♯* a semitone in 7, 19, 31, . . fifths; we thus obtain the primary negative systems, or by depressing *d* two semitones, by which we get the secondary negative systems,—and so on.

## FORMATION OF MAJOR THIRDS.

### 7. NEGATIVE SYSTEMS.

The departure of the perfect third is $-\cdot13686$, as we have seen (section 2); that is to say, it falls short of the E. T. third by this fraction of an E. T. semitone. Hence negative systems, where the fifth is of the form $7-\delta$, form their thirds in accordance with the ordinary notation of music. For if, in such a system, we form a third by taking four fifths up, we have a third with negative departure $(=-4\delta)$, which can approximately represent the departure of the perfect third. Thus, *c♯* is either the third to *A*, or four fifths up from *A*, in accordance with the usage of musicians.

EXAMPLE.—In the system of 31 the departure of each fifth is $-\frac{1}{31}$, and that of the third by four-fifths up is $-\frac{4}{31} = -\cdot 12903$; and this differs from the departure of the perfect third ($= -\cdot 13686$) only by the small error $\cdot 00783$, or considerably less than the hundredth of a semitone.

## 8. Positive Systems.

Positive systems, on the other hand, form their approximately perfect thirds by 8 fifths down; for their fifths, being larger than E. T. fifths, depress the pitch below E. T. when tuned downwards. Thus, if the fifth be of the form $7+\delta$, 8 fifths down give the negative departure ($=-8\delta$), which can approximately represent the departure of the perfect third. Thus the third of A should be D♭, which is inconsistent with musical usage. Hence positive systems require a separate notation, to which we will return immediately.

EXAMPLE 1.—Regular system of perfect fifths. The departure of a perfect fifth is $\cdot 01955$, as we have seen. Eight fifths down give therefore a departure $= -8 \times \cdot 01955 = -\cdot 15640$; and this exceeds the departure of a perfect third ($= -\cdot 13686$) by the error $\cdot 01954$; a quantity which is the same, within one unit in the last place, as the departure of the perfect fifth, or the error of the E. T. fifth, which is the same thing.

EXAMPLE 2.—System of 53. Departure of third by 8 fifths down $= -\frac{8}{53} = -\cdot 15095$.

EXAMPLE 3.—System of 118. Departure of third by 8 fifths down $= -\frac{8}{59} = -\cdot 13560$, the error of which is little more than the thousandth part of a semitone.

## 9. Notation.

Helmholtz employs a peculiar notation for the system generally called by his name, which has very nearly perfect fifths and perfect thirds.* We shall speak of this system in general as the positive system of perfect thirds. Helmholtz's employment of this notation is marked by several peculiarities, which we need not here discuss; the objection that this notation is unsuitable for use with musical symbols is sufficient to warrant us in disregarding it.

The following notation is here adopted for positive systems in general: it is not intended to be limited to any one system, like Helmholtz's. In fact, as it consists entirely of an indication of position in a series of fifths, it may, when desired, be used with negative systems.

The notes are arranged in series, in order of successive fifths. Each series contains twelve fifths, from *f*♯ up to *b*. The series *f*♯—*b* is called the unmarked series; it contains the standard, or unmarked *c*. Each note of the next series of twelve fifths up is affected with the mark (⁄), which is called a mark of elevation,

* See note at p. 121, *post*.

and is drawn upwards in the direction of writing. The next series of twelve fifths up is affected with the mark (//); and the succeeding series of twelve fifths up are affected with a number of marks of elevation corresponding to their position, (///), (////), and so on. The series below the unmarked series is affected with the mark (\\), which is called a mark of depression, and is drawn downwards in the direction of writing; and the succeeding series, in a descending order of fifths, are affected with a number of marks of depression corresponding to their position, (\\\\), (\\\\\\), and so on. Such fifths as *\\b–f♯*, *b–/f♯*, which join any two of the series of the notation, have the same value as all the rest.

Thus, for example, the interval *c–/c* is the departure of twelve fifths. *c–\\e* are related through eight fifths downward from *c*. Hence in positive systems *\\e* is the note which forms an approximately perfect third with *c*.

N.B.—It is to be noted that the position in the series of fifths is determined only by the notation here introduced; *i.e.*, *c♯* and *d♭* mean exactly the same thing, and refer only to one of the twelve E. T. divisions of the octave. Regarded as belonging to an assigned system, *c♯* or *d♭* would mean that note of the unmarked series which is five fifths below the unmarked or standard *c*.

## 10. Rule for Thirds in Positive Systems.

If we write down one of the series of the notation:—

*f♯–c♯–g♯–d♯–a♯–f–c–g–d–a–e–b*,

and remember that positive systems form their thirds by eight fifths down, we have the rule:—

The four accidentals on the left in any series of the notation form major thirds to the four notes on the right of the same series. All other notes have their major thirds in the next series below. Thus, *d–f♯* and *c–\\e* are major thirds.

## 11. Employment of the Notation in Music.

This notation is suitable for employment with written music. Its appearance will be generally taken to indicate the employment of a system with perfect or approximately perfect fifths, unless anything is said to the contrary.

The following passage is an example:—

The interval *g*– *f* is a close approximation to the harmonic or natural seventh; ⁄*a*♭–*f*♯ is rendered very smooth by the employment of the same interval. The development of the practical use of the notation is deferred for the present.

The notation is also useful for the discussion of some systems of historical interest. Thus, we have a scale of F in Mersenne, whose work bears the date 1636, with eighteen notes to the octave. This possessed the following resources:—

Major chords of *c*–*f*–*b*♭–*e*♭,
„ ╲*e*–╲*a*–╲*d*–╲*g*, thirds to the above.
„ ⁄*a*♭–⁄*d*♭–⁄*g*♭, thirds below *c*–*f*–*b*♭.

Minor chords of *c*–*f*–*b*♭–*e*♭,
„ ╲*e*–╲*a*–╲*d*–╲*g*, thirds to the above.
„ ╲*c*♯–╲*f*♯–╲╲*b*, thirds to ╲*a*–╲*d*–╲*g*.

We have here the two forms of second of the key, *g* and ╲*g*, differing approximately by a comma. This double form appears in all good attempts at systems with perfect fifths.

## 12. The 'Generalised Keyboard.'

A keyboard has been designed and constructed, by means of which the notes of all regular systems, positive and negative, can be brought under the control of the fingers. The detailed account of this keyboard is deferred for the present. It contains eighty-four keys in every octave; the instrument of which it forms a part is a harmonium, and the system of tuning is that which divides the octave into fifty-three equal intervals. The form of fingering is the same in all keys. Such passages as the example in musical type given above can be readily performed upon it.

SYMMETRICAL ARRANGEMENT OF THE NOTES OF THOMPSON'S ENHARMONIC ORGAN.

*The Subscripts 1, 2, and 3 refer to its Three Keyboards.*

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | $'c_1$ | . | . | . | . | . | . | . | . | . | . | . | $'c_1$ |
| 11 | . | . | . | . | . | $'f_1$ | . | . | . | . | . | . | . |
| 10 | . | . | . | . | . | . | . | . | . | . | $'bb_1$ | . | . |
| 9 | . | . | . | $'eb_1$ | . | . | . | . | . | . | . | . | . |
| 8 | . | . | . | . | . | . | . | . | $'ab_1$ | . | . | . | . |
| 7 | . | $'c\sharp_1$ | . | . | . | . | . | . | . | . | . | . | . |
| 6 | . | . | . | . | . | . | $'f\sharp_1$ | . | . | . | . | . | . |
| 5 | | | | | | | | | | | | | |
| 4 | . | . | . | . | $e_2$ | . | . | . | . | . | . | . | . |
| 3 | . | . | . | . | . | . | . | . | . | $a_{1\cdot2}$ | . | . | . |
| 2 | . | . | $d_{1\cdot2}$ | . | . | . | . | . | . | . | . | . | . |
| 1 | . | . | . | . | . | . | . | $g_{1\cdot2\cdot3}$ | . | . | . | . | . |
| 12 | $c_{1\cdot2\cdot3}$ | . | . | . | . | . | . | . | . | . | . | . | $c_{1\cdot2\cdot3}$ |
| 11 | . | . | . | . | . | $f_{1\cdot2\cdot3}$ | . | . | . | . | . | . | . |
| 10 | . | . | . | . | . | . | . | . | . | . | $bb_{1\cdot3}$ | . | . |
| 9 | . | . | . | $eb_{1\cdot3}$ | . | . | . | . | . | . | . | . | . |
| 8 | . | . | . | . | . | . | . | . | $ab_3$ | . | . | . | . |
| 7 | . | $c\sharp_2$ | . | . | . | . | . | . | . | . | . | . | . |
| 6 | . | . | . | . | . | . | $f\sharp_{1\cdot2}$ | . | . | . | . | . | . |
| 5 | . | . | . | . | . | . | . | . | . | . | . | $`b_{1\cdot2\cdot3}$ | . |
| 4 | . | . | . | . | $`e_{1\cdot2\cdot3}$ | . | . | . | . | . | . | . | . |
| 3 | . | . | . | . | . | . | . | . | . | $`a_{1\cdot2\cdot3}$ | . | . | . |
| 2 | . | . | $`d_{1\cdot2\cdot3}$ | . | . | . | . | . | . | . | . | . | . |
| 1 | . | . | . | . | . | . | . | $`g_{1\cdot2\cdot3}$ | . | . | . | . | . |
| 12 | $`c_3$ | . | . | . | . | . | . | . | . | . | . | . | $`c_3$ |
| 11 | . | . | . | . | . | $`f_3$ | . | . | . | . | . | . | . |
| 10 | . | . | . | . | . | . | . | . | . | . | $`bb_2$ | . | . |
| 9 | . | . | . | $`eb_2$ | . | . | . | . | . | . | . | . | . |
| 8 | . | . | . | . | . | . | . | . | $`ab_{2\cdot3}$ | . | . | . | . |
| 7 | . | $`c\sharp_{2\cdot3}$ | . | . | . | . | . | . | . | . | . | . | . |
| 6 | . | . | . | . | . | . | $`f\sharp_{2\cdot3}$ | . | . | . | . | . | . |
| 5 | . | . | . | . | . | . | . | . | . | . | . | $``b_{2\cdot3}$ | . |
| 4 | . | . | . | . | $``e_3$ | . | . | . | . | . | . | . | . |
| 3 | . | . | . | . | . | . | . | . | . | $``a_3$ | . | . | . |
| 2 | | | | | | | | | | | | | |
| 1 | . | . | . | . | . | . | . | $``g_2$ | . | . | . | . | . |
| 12 | $``c_2$ | . | . | . | . | . | . | . | . | . | . | . | $``c_2$ |
| 11 | . | . | . | . | . | $``f_3$ | . | . | . | . | . | . | . |
| 10 | . | . | . | . | . | . | . | . | . | . | $``bb_2$ | . | . |
| 9 | . | . | . | $``eb_2$ | . | . | . | . | . | . | . | . | . |
| 8 | . | . | . | . | . | . | . | . | $``ab_3$ | . | . | . | . |
| 7 | . | $``c\sharp_3$ | . | . | . | . | . | . | . | . | . | . | . |

### 13. Principle of Symmetrical Arrangement.

This principle is employed in the design of the above-mentioned keyboard; and it is owing to its properties that the fingering is the same in all keys. It may be thus stated:—

If we place the E. T. notes on a horizontal line in the order of the scale, and set off the departures of the notes of any system at right angles to the E. T. line, sharp departures up and flat departures down, we obtain the positions of a symmetrical arrangement. The accompanying table is a symmetrical arrangement of the notes of General Thompson's enharmonic organ. The following series of intervals lie on characteristically-placed straight lines in a symmetrical arrangement:—

| Name of interval in general case. | Name when fifths are perfect. | Series of intervals. |
|---|---|---|
| 2-fifths tone . . . . | Major tone . . . . . | *c–d–e* . . . |
| 5-fifths semitone . . . | Pythagorean semitone. . | */c–/c♯–d–e♭–\e* . . |
| 7-fifths semitone . . . | Apotomè Pythagorica . . | *\c–c♯–d–/e♭* . . . |
| Third by 4 fifths up . . | Dissonant, or Pythagorean third . . . . . | *c–e–/a♭–/c* . . . |
| Third by 8 fifths down . | (Approximately perfect third.) | *c–\e–\a♭–\\c* . . |

The departure of twelve perfect fifths, or the Pythagorean comma, $= 12 \times \cdot 01955 = \cdot 23460$. The ordinary comma of $(\frac{81}{80})$ is ·21506. For a certain degree of approximation we can neglect the difference between these quantities, and speak of a comma without specialising our meaning. In this sense we may read the last of the above series of intervals as showing us that three perfect thirds fall short of the octave by two commas approximately. We may return to this point hereafter.

In the symmetrical arrangement of the notes of General Thompson's organ, we may note specially, first, the effect of the distribution of the notes over these keyboards. For instance, the notes of the chord of A minor are all present ($a_{1 \cdot 2}$–$/c_{1}.$–$e_{2}.$); but the third and fifth are on different keyboards, so that the chord would not be generally available.

Again, the notes *b* and *\\d* are missing from the otherwise complete scheme; we notice the number of chords which their absence destroys.

In the present paper the endeavour has been made to present to the members of the Musical Association the more fundamental portions of the theory of the subject. It is hoped that this treatment may facilitate the comprehension of such historical points and such further developments as may be hereafter brought before the Association.