FAVORED CARDINALITIES OF SCALES

PETER BUCH

ABSTRACT. In Western music history, the musical scales that were the most successful have 5, 7, or 12 tones per octave, respectively. Experimentalists have used 19, 31, or more tones per octave. A novel calculation is presented that yields these cardinalities, with a minimum of assumptions about scales, and without imposing a p-limit. The Riemann zeta function plays a central role.

INTRODUCTION

Scales are an important structural element of music. From prehistoric times until now, as music developed, the scales in use developed in parallel. And there is a long history of mathematical analysis of scales, starting with Pythagoras, and with contributions of Avicenna (Ibn Sina), Zarlino, and hundreds of others in the modern times.

Main stations of scale development are: the pentatonic scale, which is still today the basic harmonic structure of very simple songs; the diatonic scale, which is what Western people sing when asked to sing a scale; the 12-tone equal-temperament (12tET) scale, which is the built-in scale of all modern keyboard instruments. The number of steps per octave – the cardinality – of these three scales are 5, 7, or 12, respectively. Beside these, microtonal scales with higher cardinalities have been used by experimentalists and studied by theorists. Cardinalities of 19, 31, and 43[1] have gathered special attention.

The manifold and profound work done on this topic (see e. g. the extensive bibliography given at [1]) cannot be discussed here in detail. From the proportions of publications as well as from the historic development it can be deduced that some cardinalities seem to be "better" than others. In this article I will introduce a new measure for how "good" a cardinality is. Conversely, this measure will allow to search for "good" cardinalities in a simple systematic way. The cardinalities favored by the majority of authors and by history will be reproduced to a high degree.

Most of the literature mentioned deals with tuning, temperament, or intonation. This article is about cardinality. Though the question

[1] These are the numbers N $(12 < N \leq 72)$ for which a Google search for "N-tone scale" gave more than 100 different results.

of cardinality and the question of tuning are closely related, only the former is studied here, while the latter is left open.

This is a short outline of my article: In section 1, I sketch the well-known basic concepts about scales; I define some notions and introduce the basic problem. Section 2 presents something that may be called the "standard theory" of mathematical scale construction in the so-called 3-limit and 5-limit. The few examples in that section tell nothing new about scale theory. The purpose is to introduce the formalism that will be used in the following. In section 3, I expand the "standard problem", and make a novel mathematical approach for solving it. The result is a list of "good" cardinalities.

This article is interdisciplinary in the sense that it treats a music theoretical topic with mathematical methods. Sections 1 and 2 will bring nothing new to music theorists, although some of the well-known facts are presented in an unusual way. Then section 3 will introduce some kind of mathematics unfamiliar to music theorists. On the other hand, mathematicians might welcome the short introduction to the topic in the first sections, whereas the last section might appear rather trivial to them.

1. Scales and numbers

In this introductory section the well-known basic concepts about scales shall be sketched, and some notions defined. For a profound introduction, see e. g. Benson [2], chapters 4–6.

1.1. **Basic notions and definitions.**

Music is a matter of individual sensation and judgement. For this reason it cannot be forced into a rigid mathematical formalism. There is no fixed set of properties that are mandatory for a musical scale; but at least there are some properties that tend to make a scale more pleasant. Thus in the following some notions are defined in a very general way, and not by rigorous mathematical definitions. They are supposed to represent the basics of the common-sense understanding of musical scales. The properties defined for intervals and scales are of a gradual nature, they may not only apply or not apply, but also may apply to some higher or lower degree. Despite of their lack of rigor, the definitions will turn out to be sufficient for deriving a result by mathematical means. The validity of the result is even underlined by the generality of this approach.

Definition 1. *A **scale** is an ordered set of pitches. It may have a periodically repeated pattern (e. g. every octave), or may not.*

Most studies on scale theory assume that scales have an octave periodicity. But this property is not needed in this article.

Definition 2. ***Evenness*** *of a scale shall denote its closeness to a scale with all steps of equal length.*

For example, an equal-temperament 12-tone scale has the greatest possible evenness. A diatonic scale in just temperament is less even, as its step lengths vary from 15:16 (112 cents[2]) to 8:9 (204 cents).

There are two reasons to aim for evenness. First – For the richness of music, scales shall have a sufficient number of different tones. And for being perceived as different, even in short notes, consecutive tones of a scale shall pitches that are not too close. The two aims of tone multitude and tone discriminability meet in the aim of evenness. Second – Transposing pieces on fixed-pitch instruments is greatly facilitated with even scales. This was the main reason for inventing 12tET.

Definition 3. *An interval of two tones is* **consonant** *if their frequencies have a ratio made up of small integer numbers.*

Lower degrees of consonance are present if the ratio contains greater integers, or if a consonant interval is detuned a little. Though this classical definition of consonance has its limitations, there is no other definition that is as widely accepted.

Definition 4. ***Harmony*** *of a scale shall mean that many pairs of its tones form consonant intervals, and that at least the simplest consonant intervals appear in the scale.*

Lower degrees of harmony are present if the intervals are less consonant, or if the number of consonant pairs is low, or if simple consonant intervals (octaves, fifths, fourths, major and minor thirds) are missing. Note that this definition does not refer to intervals generating a scale, but to intervals actually appearing in it. The scales considered here need not be generated by a set of intervals.

Certainly it is possible to define numerical measures for evenness, consonance, and harmony. But this is not done as it is not necessary here, and thus the problem is avoided if such measures really represent the human sensation about these notions.

1.2. **The scale construction problem.** It is a general aim to construct scales that are both even and harmonic. But the two demands of evenness and harmony cannot be fulfilled perfectly at the same time. A just-tempered scale contains a lot of consonant intervals (and some dissonant ones, which cannot be avoided completely – e. g. "wolf intervals") and thus is well harmonic, but its steps are only approximately (albeit sufficiently) equal. An equal-tempered scale has perfectly equal steps, but the intervals it contains can only be approximately consonant. This is because, e. g., octaves (frequency ratio 1:2) and fifths

[2] 1 cent is defined as an interval length such that a perfect octave has 1200 cents.

(frequency ratio 2:3) are incommensurate: One cannot find a common measure, which is a smaller "basic interval" 1:b (with a real number $b > 1$), and two integers n, m, such that the octave has the length of n basic intervals (steps) and the fifth the length of m steps, i. e.

$$2 = b^n \qquad \text{and} \qquad \frac{3}{2} = b^m. \tag{1}$$

Thus, constructing an even and harmonic scale means finding a compromise between the two demands of harmony and evenness, and finding *approximations* to equations like eq. 1. There are (at least) two different ways of construction:

The first is an approach originating from harmony: Starting with an empty set, tones are added one after another making sure that every new tone forms consonant intervals with as many of the previous tones as possible. This can be formalized by using a set of generating intervals. The procedure is stopped when an approximately "even" series of tones is achieved. This is likely to be the way scales developed in history. In modern times, this procedure has been refined by group theoretical considerations [3–5].

The second way starts from evenness: A basic step interval 1:b is taken to build a perfectly even scale with the frequency ratios

$$\cdots : b^{-1} : 1 : b : b^2 : b^3 : \cdots \tag{2}$$

If b is chosen luckily, this scale may contain a lot of consonant and almost consonant intervals and thus be more or less harmonic.

These two construction schemes are equivalent in the following sense. If, in a harmonically constructed scale, all steps are replaced by an average-length step, this will be a marginal change, as the scale was approximately even already before. This means that the resulting scale has the form of eq. 2 and is still approximately harmonic. Conversely, starting from an equally-spaced scale with approximate harmony, the tones may be shifted a small amount in order to get consonant intervals. So there is a correspondence between scales with higher harmony and scales with higher evenness. A common characteristic of such corresponding scales is the *average* step length.

Definition 5. *If a scale is sufficiently harmonic to contain octaves, or intervals very close to an octave (pseudo-octaves), and if it is sufficiently even that every (pseudo-)octave is divided into the same number of steps N, then N is called the **cardinality** of the scale.*

Clearly this definition applies only to a subset of all possible scales. For all other scales, N remains undefined. N (if it exists) is the average length of the (pseudo-)octaves divided by the average step length. So corresponding scales are characterized by their common cardinality (if they have one, by definition 5). The difference between scales with the same cardinality is their *tuning*. Tuning theory is in principle the theory of finding compromises between harmony and evenness. In the rest of this article, no preference will be given either to harmony or to evenness.

In the history of music, many scales have been constructed, whether by intuition or by calculation. The most familiar are the pentatonic, the diatonic, and the chromatic scale with a cardinality of 5, 7, or 12, respectively; but also microtonal scales [2, ch. 6] with 19, 31 or more steps per octave have been studied and eventually used.

The aim of this article is to investigate which the preferred cardinalities of scales are, regardless of the specific tuning of the scales. The reasoning is solely based on the assumption that the scales shall be even and harmonic as defined above.

2. Solutions for small factors

2.1. **Factors 2 and 3.** Equation system 1 can be rewritten as

$$\frac{q_2}{\log 2} = \frac{q_3}{\log 3} \quad \left(= \frac{1}{\log b} \right) \tag{3}$$

with integers q_2 $(=n)$ and q_3 $(=m+n)$. $1{:}b$ is the average step interval. The logarithms may be taken to any base, since a change in the base makes the terms change by a common factor. Frequently a base of 2 is taken making $\log 2 = 1$ and thus measuring intervals in units of octaves. But octaves shall not be treated in a preferred way here. In the following, log shall denote the natural logarithm (base e). The goal is to find solutions q_2, q_3. With such a solution, $1/\log b$ will arise. For this reason the rightmost side of the equation is put in parentheses.

Note that there are no (integer) solutions to eq. 3 . But there are approximations in the shape of "near-integer" solutions.

One such solution is given by

$$q_2 = 7 \quad , \quad q_3 = 11.0947375\ldots \approx 11 \tag{4}$$

It tells that a scale can be constructed with 7 steps per octave and, on average, q_3–q_2=4.0947375... steps per perfect fifth. A number of 4-step intervals in this scale can be made perfect fifths, but not all. As 4 steps are, on average again, too short for a perfect fifth, there must be 4-step intervals (at least one per octave) that are shorter (flatter).

For example, consider the well-known 7-step tuning

$$\begin{array}{cccccccccccccc} \cdots : & \frac{5}{6} : & \frac{15}{16} : & 1 : & \frac{9}{8} : & \frac{5}{4} : & \frac{4}{3} : & \frac{3}{2} : & \frac{5}{3} : & \frac{15}{8} : & 2 : & \frac{9}{4} : & \frac{5}{2} : & \cdots \\ \cdots & \text{A} & \text{B} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{a} & \text{b} & \text{c'} & \text{d'} & \text{e'} & \cdots \end{array}$$

(This shall only serve as an example, and the way such ratios are found shall not be discussed here.) This scale is octave periodic, every 7-step interval is a perfect octave. There are many perfect fifths in it, but also inevitable shorter 4-step intervals, such as B–f (45:64, diminished fifth) and d–a (27:40, comma flat fifth).

Another scale based on solution 4 is an equally spaced scale as in eq. 2 with $b = \sqrt[7]{2}$. Here every fifth is flatter than perfect.

Another solution to eq. 3 is

$$q_2 = 6.8380452\ldots \approx 7 \quad , \quad q_3 = 10.8380452\ldots \approx 11 \tag{5}$$

It represents the same near-integer approximation as eq. 4, but is "tuned" to perfect fifths, as $q_3 - q_2 = 4$ here. By this it is possible to construct a scale with all fifths perfect. Now, as $q_2 < 7$, at least some of the octaves in such a scale must be sharper than perfect.

The last two examples were rather unusual. They demonstrate the generality of the approach in this article.

Better than the previous approximations to eq. 3 is the well known near-integer solution

$$q_2 = 12 \quad , \quad q_3 = 19.0195500\ldots \approx 19 \tag{6}$$

With a basic interval of $1{:}b = 1{:}\ \sqrt[12]{2}$ we get an equal-step scale (eq. 2), where 12 steps form an octave and 7 steps form an approximate fifth (because 7.0195500... steps are a perfect fifth). This is the familiar 12tET scale.

A series of approximations to eq. 3 with increasing precision can be found by the continued fraction expansion of $\log 3/\log 2$ [2, ch. 5].

If a scale is based on a solution of eq. 3, and if q_2 is a near-integer, the scale will have a well defined cardinality

$$N = nint(q_2) \quad , \tag{7}$$

where $nint(x)$ is the nearest-integer function, which maps a real number x to the nearest integer[3].

2.2. **More factors.** For high degrees of harmony (see definition 4), scales will have to contain more kinds of consonant intervals, e. g. fourths (3:4), major thirds (4:5), and minor thirds (5:6). In order to take these into account, eq. 3 has to be extended to these larger factors:

$$\frac{q_2}{\log 2} = \frac{q_3}{\log 3} = \frac{q_4}{\log 4} = \frac{q_5}{\log 5} = \frac{q_6}{\log 6} \tag{8}$$

Given near-integers q_2, q_3 that satisfy eq. 8, there are near-integers $q_4{=}2q_2$ and $q_6{=}q_2{+}q_3$ that do so too. The equation system could be restricted to prime numbers before searching approximations. But this

[3] The definition of $nint(x)$ in the middle between integers is of no interest here, because q_2 is a near-integer.

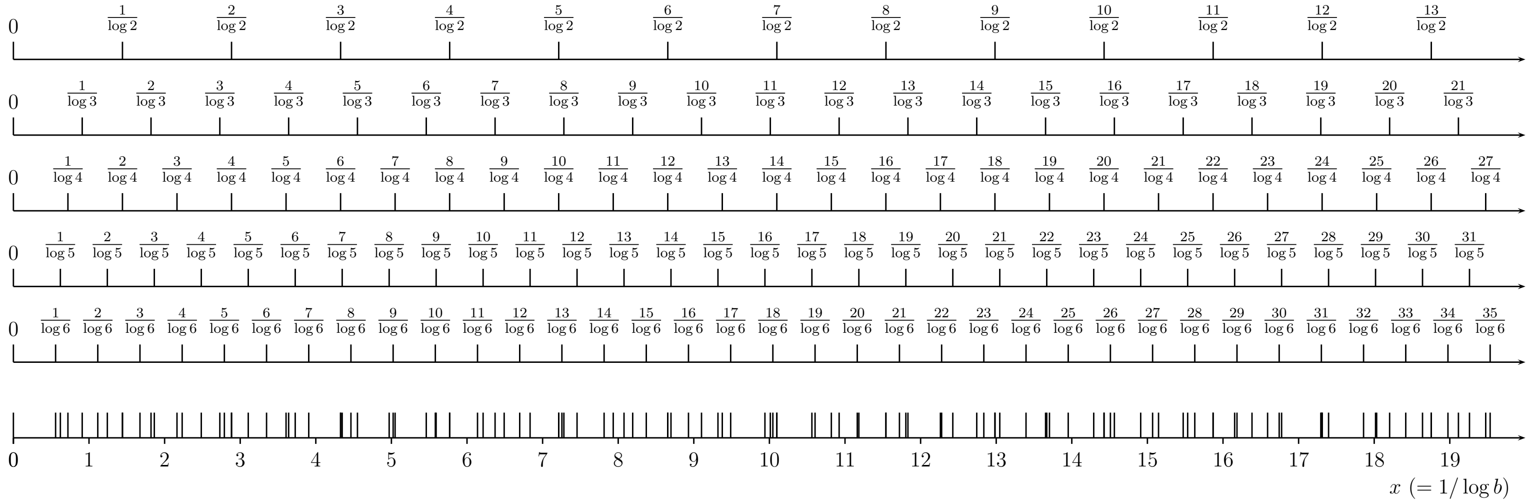

FIGURE 1. Search for approximate solutions of eq. 8

is not necessary. Moreover, leaving the composite integers in the equations will give a higher representation to smaller primes, as they appear as factors more frequently.

In order to find approximate solutions for eq. 8 "visually", the numbers $1/\log 2$, $2/\log 2$, $3/\log 2$, $4/\log 2$ ... are marked on the axis of positive real numbers x, the numbers $1/\log 3$, $2/\log 3$, $3/\log 3$, $4/\log 3$ ... on a second copy of the axis, the numbers $1/\log 4$, $2/\log 4$, $3/\log 4$, $4/\log 4$ ... on a third copy of the axis, and so on, see fig. 1.

The values of x, for which marks on the different axes nearly coincide, correspond to near-integer solutions q_n to eq. 8. So the 12th mark on the $\log 2$ axis is very close to the 19th mark on the $\log 3$ axis (cf. eq. 6), the 24th mark on the $\log 4$ axis (of course), the 28th mark on the $\log 5$ axis, and the 31st mark on the $\log 6$ axis, all of them near $x \approx 17.3$. This describes a scale, where the frequency ratios of 1:2, 1:3, 1:4, 1:5, 1:6 are represented by 12, 19, 24, 28, 31 steps, respectively. One step corresponds, at least on average, to the ratio 1:b with $1/\log b = x$. This means 1:$b \approx$ 1:1.059, a chromatic semitone. So the 12tET scale is not only based on a solution of eq. 3, but also on a solution of equation system 8.

Now, how can all such coincidences of axis marks be found, in a more systematic way than just by eye? As a first step, in order to allow for near-integers, the marks of fig. 1 will be replaced by some kind of fuzzy marks. This is done by a function that marks all integers in a more tolerant or diffuse way, a not yet further specified periodic function $f(x)$ that has "peaks" of some width at every integer-valued x.

The x-scaled functions $f(x \log 2)$, $f(x \log 3)$, $f(x \log 4)$, $f(x \log 5)$, $f(x \log 6)$ are displayed in fig. 2 and have peaks at the marks of fig. 1. The function at the bottom of fig. 2 is the sum of the above functions,

$$F(x) = \sum_{n=2}^{6} f(x \log n) \tag{9}$$

At any x where peaks of different functions $f(x \log n)$ coincide within the peak width, there is an especially great value of the sum function $F(x)$. The more peaks coincide, and the closer the coincidence is, the greater $F(x)$ will be. So the procedure for finding good coincidences is finding local maxima of the "coincidence function" $F(x)$, which are – in some way – significantly high.

Every peak of $f(x \log 2)$ is accompanied by a peak of $f(x \log 4)$, which is half as wide, and wherever peaks of $f(x \log 2)$ and $f(x \log 3)$ coincide, there is a narrower peak of $f(x \log 6)$ with them. In this way peaks of $F(x)$ involving small prime numbers are higher and narrower than others. This would not be the case if the composite factors had been eliminated from eq. 8. It is a desired effect as it reflects the higher importance of the simpler consonant intervals in a scale.

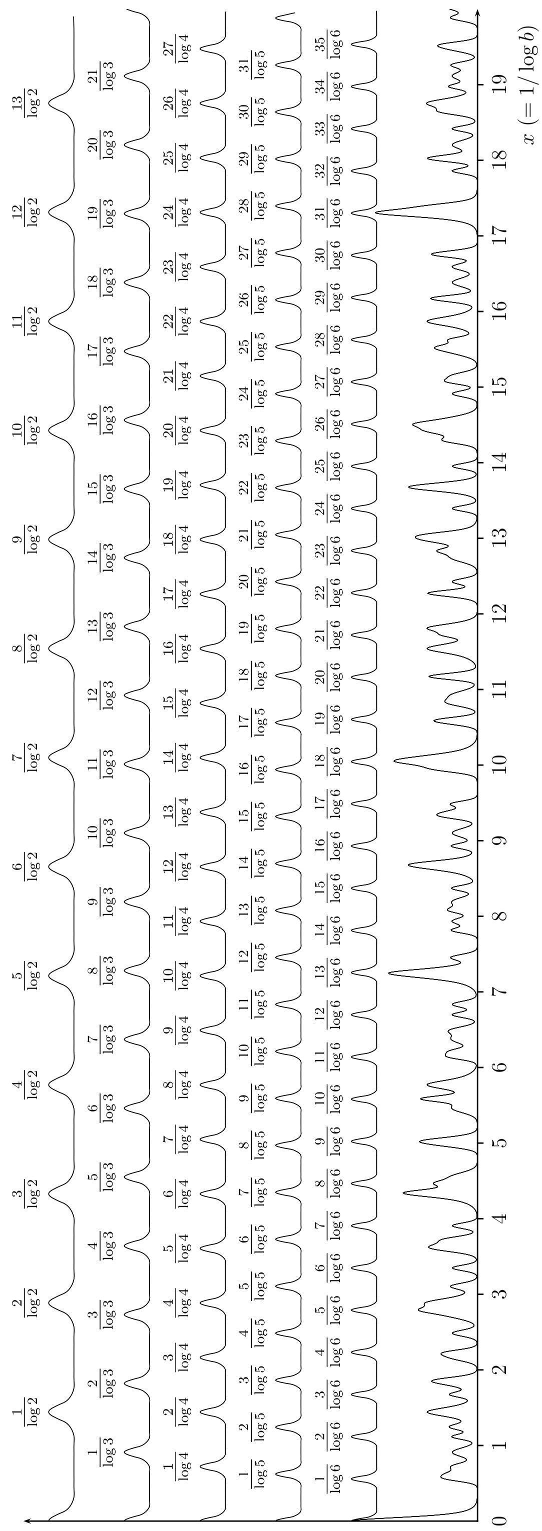

FIGURE 2. "Fuzzy" version of figure 1

3. A new mathematical approach

3.1. **Giving up any p-limit.** A scale or a set of intervals is called to be in the p-limit, if p is the greatest prime number appearing in the frequency ratios. So equation 3 represents the 3-limit, and equation 8 the 5-limit. Now the next step is to give up any artificial p-limit and extend our calculation to all frequency ratios 1:n. The goal is to find some x such that

$$x = \frac{q_n}{\log n} \quad , \quad n = 2, 3, 4, 5, 6, 7, 8, \dots \tag{10}$$

with appropriate near-integers q_n.

[4] A suitable coincidence function is

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} f(x \log n) \tag{11}$$

The new n=1 term of the sum can safely be added because it is a constant – $f(0)$ – and does not change the positions of the local maxima of $F(x)$. The factor $1/n^\sigma$ is new as well (unless $\sigma = 0$) and has, with a positive σ, a twofold purpose: first it can ensure convergence of the infinite sum; second, it gives less weight to greater n. The latter effect can be regarded as a gentle replacement for the harsh cut-off by a p-limit.

As an integer-marking function is taken

$$f(x) = \cos 2\pi x \tag{12}$$

which has peaks $f(x) = 1$ for all integer-valued x. It seems to be a very poor integer-marking function, because of the wide "peaks" of the simple cosine function. This will be discussed later.

With this $f(x)$, the coincidence function can be transformed as follows:

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \cos(2\pi x \log n) \\ &= \sum_{n=1}^{\infty} n^{-\sigma} \operatorname{Re} e^{-2\pi i x \log n} \\ &= \sum_{n=1}^{\infty} n^{-\sigma} \operatorname{Re} n^{-2\pi i x} \\ &= \sum_{n=1}^{\infty} \operatorname{Re} n^{-(\sigma + 2\pi i x)} \\ &= \operatorname{Re} \zeta(\sigma + 2\pi i x) \end{aligned} \tag{13}$$

[4] Readers who are not familiar with the following mathematics may skip one page of text and continue at the paragraph marked by an asterisk. The function $F(x)$ of section 2 will just be replaced by a different $F(x)$.

Here Re denotes the real part of a complex number and ζ is the complex-valued Riemann zeta function[5]. Whereas the infinite sum converges only for $\sigma > 1$, the zeta function can be extended analytically to all real σ. The zeta function has been investigated intensively on the so-called critical line, which is given by the special value $\sigma = 1/2$. For an introduction to the zeta function, see e. g. Bombieri [7].

* The problem of finding coincidences in the $q_n/\log n$ $(n = 2, 3, 4, 5, ...)$ is now translated into the simpler task of finding local maxima of $\mathrm{Re}\,\zeta(\sigma + 2\pi ix)$ as a function of x, with a fixed σ. Fig. 3 shows $F(x)$ and the axes of fig. 1 for $n = 2, 3, 5$. The correspondence between local maxima and coincidences can be seen easily. A value of 1/2 for σ has been chosen just for convenience, but it can be shown that the position of the very high local maxima is almost the same for a wide range of σ values.

The low quality of the cosine function as an integer marker, as noted above, is amended by the fact that $F(x)$ is a sum over all integers, not only primes: thus there is not only a term for $n = 2$, but also for the powers of 2, $n = 4, 8, 16, ...$, which have narrower maxima which add up at the maxima of the $n = 2$ term and cancel out elsewhere. By this the $n = 2$ maxima are "sharpened".

3.2. **Finding good approximations.** Now what are the significant local maxima of $F(x)$? A look at the function shows that the peaks tend to get higher for greater (positive) x, with a decreasing slope. That is why significance should not be defined by a constant threshold.

For this reason the heights of the local maxima are measured relative to the dotted line in fig. 3. The dotted line ("growth curve") is an empirically found smooth function that approximates the growth of those local maxima that are greater than the ones before them.[6] A local maximum shall be considered significant if it reaches 95% of the growth curve. This is an arbitrary threshold value for the time being, but it will turn out to give sensible results.

Every "high" local maximum of the zeta function is located very close to one of the so-called Gram points[7] g_k $(k = -1, 0, 1, 2 \ldots)$[8]. The Gram points (scaled down by a factor of 2π to match the definition

[5] Named after Bernhard Riemann (1826–1866)[6], mathematician, not to be mistaken for Hugo Riemann (1849–1919), musicologist. The zeta function was first introduced by Leonhard Euler (1707–1783), who also invented the *Tonnetz*, but was unaware of the connection between zeta function and music theory.
So, Riemann $\neq$ Riemann, but Euler = Euler.

[6] The empiric growth curve is $\frac{2}{3}\exp(\sqrt{3\vartheta'(2\pi x)})$, where ϑ' is the derivative of the Riemann-Siegel theta function.

[7] After Jorgen Gram (1850-1916), Danish mathematician.
The Gram points g_k are defined by $\mathrm{Im}\,\zeta(\frac{1}{2} + ig_k) = 0$, $\mathrm{Re}\,\zeta(\frac{1}{2} + ig_k) \neq 0$

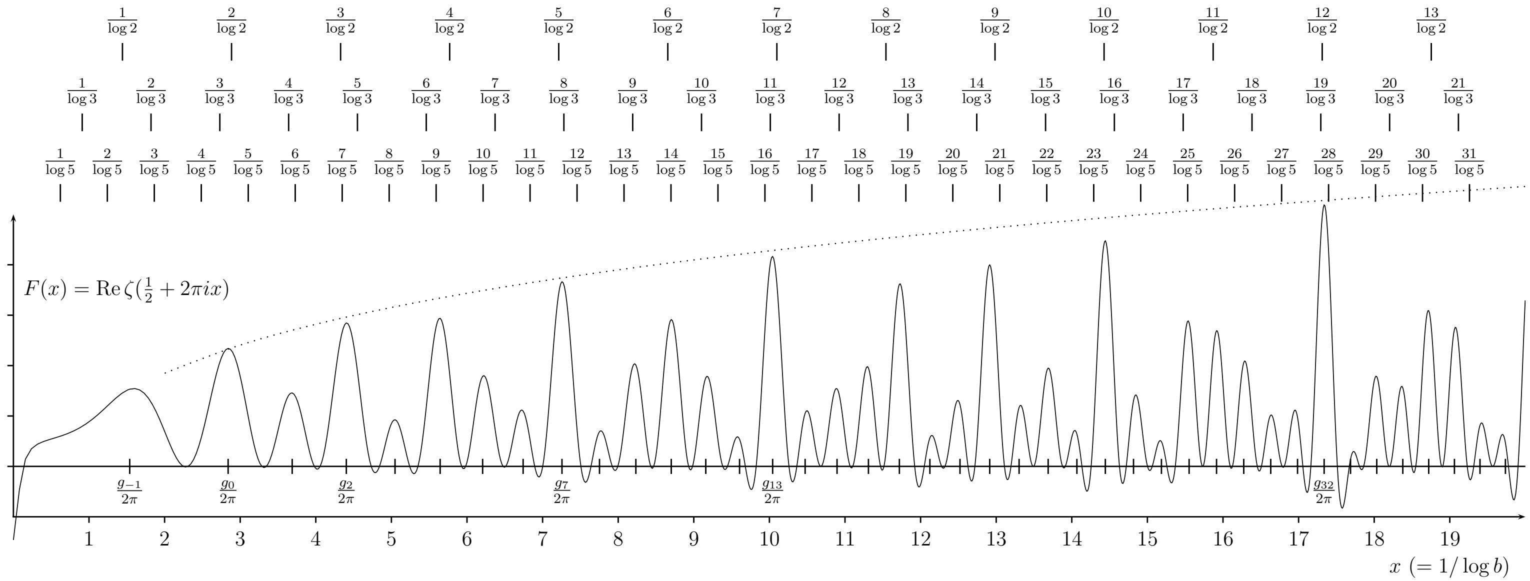

FIGURE 3. The coincidence function

k	$x = g_k/2\pi$	$F(x) = \zeta(\frac{1}{2}+ig_k)$	relative height	$q_2 = x\log 2$	$q_3 = x\log 3$	$q_5 = x\log 5$	$q_7 = x\log 7$
-1	1.54	1.53	102.89%	1.07	1.69	2.48	2.99
0	2.84	2.34	100.45%	1.97	3.12	4.57	5.53
2	4.40	2.85	96.07%	3.05	4.84	7.09	8.57
7	7.26	3.66	97.98%	5.03	7.97	11.68	14.12
13	10.04	4.16	97.22%	6.96	11.03	16.16	19.54
32	17.34	5.19	98.31%	12.02	19.05	27.90	33.74
63	27.34	5.97	96.53%	18.95	30.04	44.01	53.21
125	44.69	7.00	96.51%	30.98	49.10	71.93	86.96
182	59.13	7.57	95.68%	40.99	64.97	95.17	115.07
255	76.46	8.25	96.57%	53.00	84.00	123.06	148.78
378	103.81	9.15	98.03%	71.95	114.04	167.07	202.00
1934	389.55	13.37	100.72%	270.02	427.97	626.96	758.03
2291	448.68	13.07	95.03%	311.00	492.93	722.13	873.10
2566	493.37	13.48	95.75%	341.98	542.02	794.04	960.05
3969	712.71	14.75	95.84%	494.01	782.99	1147.06	1386.87
...	...	...	...	...	...	...	...

TABLE 1. Near-integer solutions of equation 10

of x) are marked on the axis at $F(x) = 0$ in fig. 3. Furthermore, the marks at the significant local maxima are labeled "$g_k/2\pi$". One can see that significant local maxima only appear at Gram points. The value of $F(x)$ at a significant local maximum is almost the same as $F(g_k/2\pi)$, insofar as in the resolution of fig. 3, the labeled Gram points cannot be distinguished from local maxima. So for finding the significant local maxima of $F(x)$, it is sufficient to evaluate the function at the Gram points, which are easy to calculate. In the following the role of the significant maxima of $F(x)$ is taken over by the Gram point values.

Table 1 lists all significant maxima (by the 95% definition) up to $x = 2000$. The entries up to k=32 are covered by fig. 3. The first four columns show the Gram point index k, its position and $F(x)$ as an absolute value as well as relative to the growth curve. Further columns show the near-integers q_n for the primes $n = 2, 3, 5, 7$. The greater $F(x)$ is, the closer the q_n are to integers, and the better an even and harmonic scale can be built in which $nint(q_n)$ steps represent an interval of 1:n. The table also shows that the smaller primes tend to have better approximations. Octaves are always approximated well; this has not been a mandatory precondition in our deduction, but it comes from the natural importance of the prime factor 2. The cardinalities N=$nint(q_2)$ in the table are 1, 2, 3, 5, 7, 12, 19, 31, 41, 53, 72, 270, 311, 342, and 494.

This is a short discussion of the individual rows of table 1.

k=–1, N=1: The first row represents a scale of octaves only, the only possible scale in the 2-limit. It is very regular and does not suffer of incommensurateness. But there are too few tones to make real music.

k=0, N=2: In the second row fifths are added to form approximate half-octave steps.

k=2, N=3: This approximation represents the different kinds of triads, which are very simple "scales".

k=7, N=5: With the local maximum at x=7.26 real music begins. The pentatonic scale is the first to have enough tones to form pleasant tunes.

k=13, N=7: The x=10.04 maximum tells that a cardinality of 7 can make good scales. So it is no surprise that the 7-step diatonic scale has been very successful; Western music is based on it. Some aspects have been discussed in subsection 2.1.

k=32, N=12: The x=17.34 row of table 1 represents a 12-tone scale (cf. eq. 6). The importance of this cardinality is well-known.

k=63, N=19; k=125, N=31: These two rows describe microtonal scales with 19 and 31 tones per octave, which have been studied by many authors.

k=182, N=41: Studies on 41-tone scales are very rare, what is a surprise with regard to the good quality of this approximation. 43-tone scales are mentioned much more frequently, though at N=43 $F(x)$ has a relative value of only 76.1%.

k=255, N=53: A division of the octave in 53 steps was already discussed by Isaac Newton.

k=378, N=72: The next approximation with $q_2 \approx 72$ is of special interest, as 72 is a multiple of 12. So a 72-tone even scale can be used not only for approximating just intonation to a high degree, but also for reproducing the chromatic 12-tone scale, which it contains 6 times. The 72tET scale has been explored by various authors.

$k \geq 1934$, $N \geq 270$: While 72 tones in an octave can still be well distinguished by the human ear, there is no question that 270 or more tones, as in the last four rows of table 1, are too many. These and the further, unlisted, solutions are only of mathematical interest.

Conclusion

The essence of the entire table 1 and of this article is that . . . 5, 7, 12, 19, 31,. . . steps per octave are all good approximations to the scale construction problem, whereas the numbers in between are of less quality, in a very general manner. I have obtained this result from only two simple assumptions about scales – harmony and evenness – without limiting the prime factors involved in the ratios in consonant intervals. The explicit construction of a scale has not been necessary for this. The question of how to tune the ... 5, 7, 12, 19, 31,... tones has been left open.

The cardinalities found to be good are not new, but well-known. This confirms the validity of the novel calculation presented here. To my knowledge this is the first time that evidence for good cardinalities is found without referring to particular tunings.

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Peter Buch
Deutsches Zentrum für Luft- und Raumfahrt
Heinrich-Konen-Str. 1
53227 Bonn
Deutschland
E-mail address: peter.buch@dlr.de